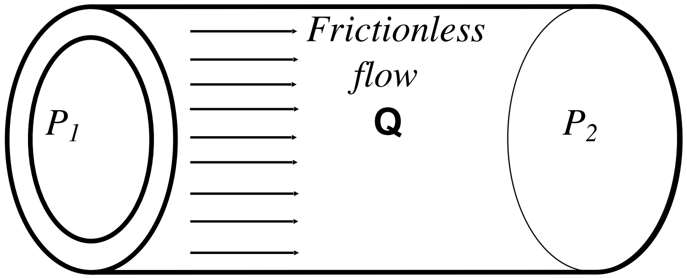
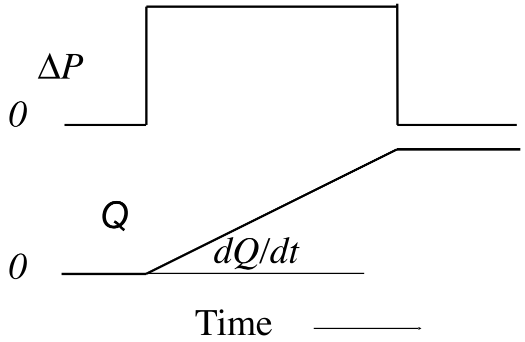
Inertance





$$\Delta P = P_1 - P_2$$

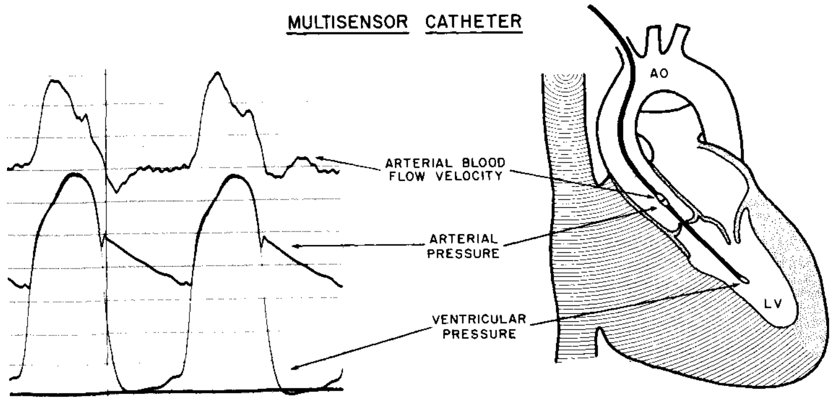
$$F = m \cdot \frac{dv}{dt}$$

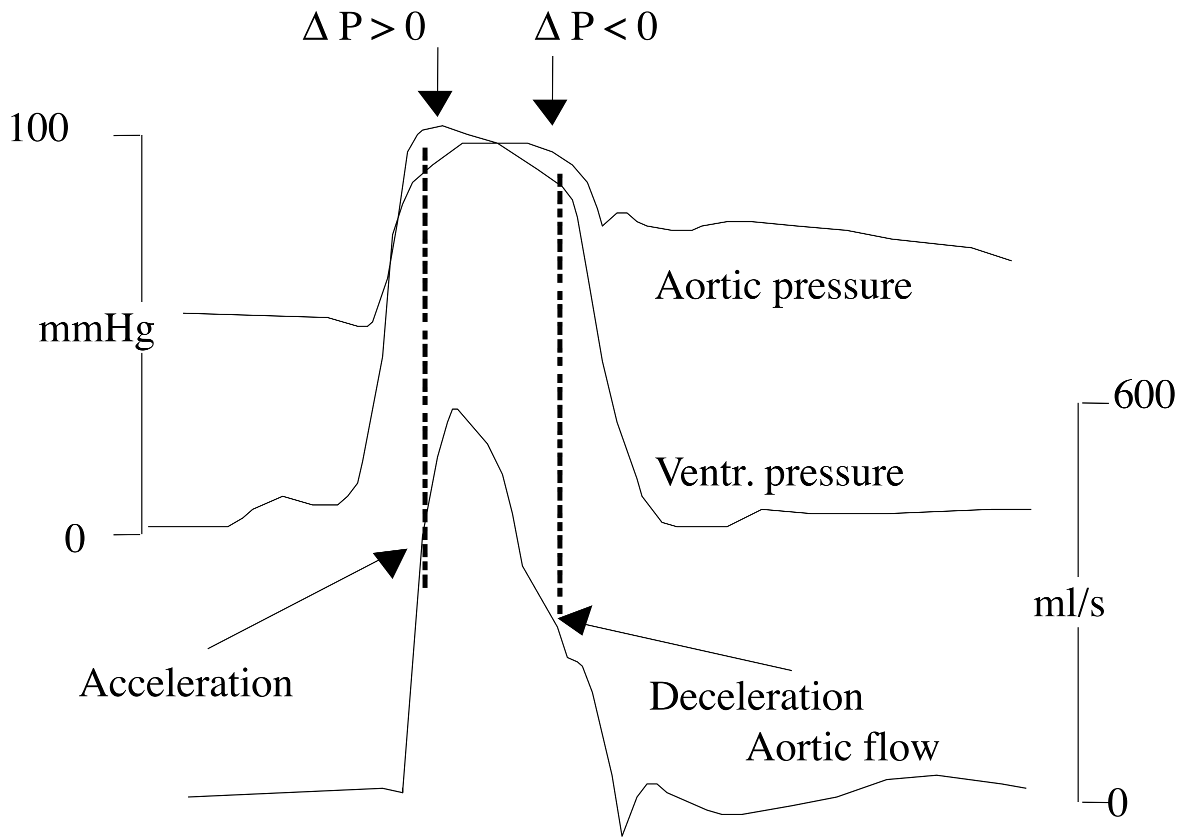
$$(P_1 - P_2)A = \rho \ell A \cdot \frac{dv}{dt} = \rho \ell \cdot \frac{dQ}{dt}$$

$$P_1 - P_2 = \frac{\rho \ell}{A} \cdot \frac{dQ}{dt} = L \cdot \frac{dQ}{dt}$$

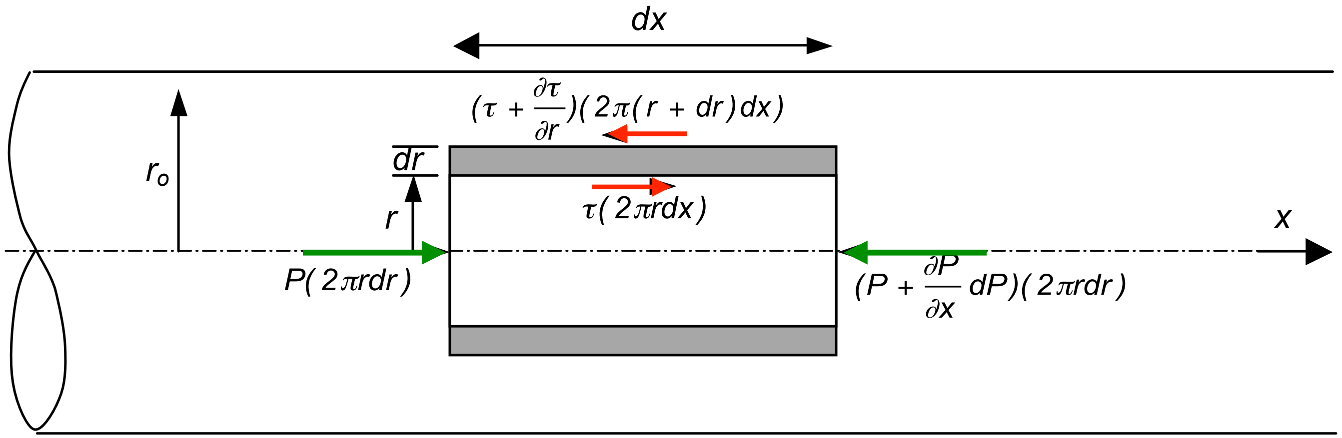
Inertance $L = \frac{\rho \ell}{A}$

Inertance in aortic flow





Womersley's theory for pulsating flow in straight rigid tubes



Newton's 2nd law: $F_x = m \cdot a_x$

$$\Rightarrow P(2\pi rdr) - (P + \frac{\partial P}{\partial x}dx)(2\pi rdr) + \tau(2\pi rdx) - (\tau + \frac{\partial \tau}{\partial r}dr)(2\pi(r+dr)dx) = \rho(2\pi rdrdx)\frac{\partial v}{\partial t}$$

higher order term

$$\Rightarrow -\frac{\partial P}{\partial x}2\pi rdrdx - \tau 2\pi drdx - \frac{\partial \tau}{\partial r}2\pi rdrdx - \frac{\partial \tau}{\partial r}2\pi dr^2dx = \rho(2\pi rdrdx)\frac{\partial v}{\partial t}$$

$$\Rightarrow -\frac{\partial P}{\partial x}r - \tau - \frac{\partial \tau}{\partial r}r = \rho r\frac{\partial v}{\partial t}$$

$$\Rightarrow -\frac{\partial P}{\partial x} - \frac{\tau}{r} - \frac{\partial \tau}{\partial r} = \rho\frac{\partial v}{\partial t} \qquad (1)$$

For Newtonian fluid: $\tau = -\mu\frac{\partial v}{\partial r}$

$$-\frac{\partial P}{\partial x}+\frac{\mu}{r}\frac{\partial v}{\partial r}+\mu\frac{\partial^2 v}{\partial r^2}=\rho\frac{\partial v}{\partial t} \qquad \Rightarrow \boxed{\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{1}{\nu}\frac{\partial v}{\partial t}=\frac{1}{\mu}\frac{\partial P}{\partial x}} \qquad (2)$$

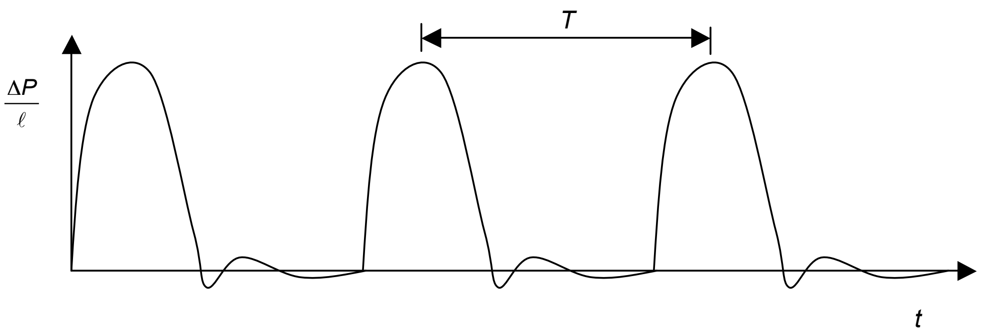
Governing equation (2):
linear partial differential equation (P.D.E.) for the velocity *v(r,t)*

where $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity of the fluid

Assume that the pressure gradient is function of time only and not a function of the radius, r: $\frac{\partial P}{\partial x}\neq f(r)$

Solution:

The governing equation (2) is linear: the general solution can be a linear superposition of other solutions. This is useful for the treatment of periodic pressure gradient functions.



Express the pressure gradient in terms of a Fourier series:

$$\frac{\Delta P}{\ell}=A_0+A_1\cos(\omega t)+B_1\sin(\omega t)+A_2\cos(2\omega t)+B_2\sin(2\omega t)+\ldots$$

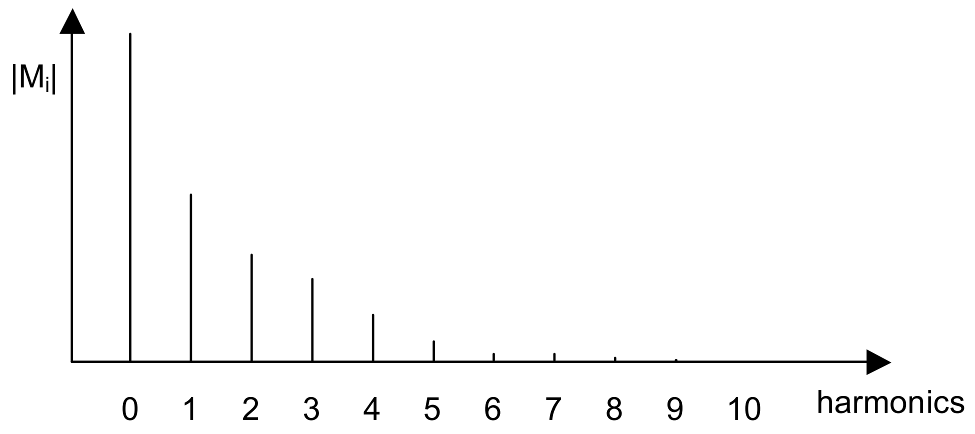
or $$\frac{\Delta P}{\ell}=M_0+M_1\cos(\omega t+\varphi_1)+M_2\cos(\omega t+\varphi_2)+\ldots$$

where $M_0=A_0$ $\quad M_i=\sqrt{A_i^2+B_i^2}$ and $\tan\varphi_i=-\frac{B_i}{A_i}$

$\omega=\frac{2\pi}{T}$ is the circular frequency

For arterial pulses:

- 5 to 10 harmonics suffice to describe the pulse.
- The amplitude of higher frequency harmonics is too small and can be neglected without introducing much error.



Strategy:

- The solution for the zero-order harmonic is obviously Poiseuille's law.
- We need to find the solution for a single harmonic pressure gradient.
- The **general solution** would be then a **linear addition of Poiseuille's solution for the zero-order term plus the solution for each harmonic**.

For a single harmonic: $$\frac{\Delta P}{\ell} = A\cos(\omega t) + B\sin(\omega t) = \mathrm{Re}[(A - iB)(\cos\omega t + i\sin\omega t)] = \mathrm{Re}[A^* e^{i\omega t}]$$

where: $A^* = A - iB$ is a complex pressure gradient

and $A^* e^{i\omega t}$ is a complex oscillatory pressure gradient

Approach: Replace $\frac{\Delta P}{\ell} = -\frac{\partial P}{\partial x}$ in the governing equation (2) by $A^* e^{i\omega t}$ and **keep the real part of the solution**

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{1}{\nu}\frac{\partial v}{\partial t}=\frac{1}{\mu}\frac{\partial P}{\partial x}$$

now becomes

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{1}{\nu}\frac{\partial v}{\partial t}=-\frac{A^*}{\mu}e^{i\omega t} \qquad (3)$$

Let us now assume that the solution to Eq. (3) is given by a complex velocity v*(r,t) of the form: $v^*(r,t)=u(r)e^{i\omega t}$

$$\frac{\partial v^*}{\partial t}=i\omega u e^{i\omega t} \qquad \frac{\partial v^*}{\partial r}=\frac{du}{dr}e^{i\omega t} \qquad \frac{\partial^2 v^*}{\partial r^2}=\frac{d^2u}{dr^2}e^{i\omega t}$$

Substituting into Eq. (3) and dividing by $e^{i\omega t}$ we obtain:

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{i\omega}{\nu}u=-\frac{A^*}{\mu} \qquad \boxed{\Rightarrow \frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{i^3\omega}{\nu}u=-\frac{A^*}{\mu}} \qquad (4)$$

linear 2nd order differential equation with a constant term on the right hand side

We first seek a general solution to the **homogeneous equation:**

$$\boxed{\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{i^3\omega}{\nu}u=0} \qquad (5)$$

General solution of Eq. (4) = General solution of homogenous Eq. (5) + Particular solution of Eq. (4)

General solution of homogenous equation $$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{i^3\omega}{\nu}u=0$$ is: $$u=C_1J_o(\lambda r)$$ where $$\lambda^2=\frac{i^3\omega}{\nu}$$

Bessel function of order 0

For the **particular solution**, we set u = C_2, and substituting into Eq. (4) $$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{i^3\omega}{\nu}u=-\frac{A^*}{\mu}$$

we obtain: $$\frac{i^3\omega C_2}{\nu}=-\frac{A^*}{\mu}$$

$$\Rightarrow C_2=-\frac{A^*}{\mu}\frac{\nu}{i^3\omega}=-\frac{A^*}{\mu}\frac{\mu}{i^3\rho\omega}=-\frac{A^*}{i^3\rho\omega}$$

The general solution becomes:

$$u(r)=C_1J_o(\lambda r)-\frac{A^*}{i^3\rho\omega} \qquad (6)$$

The constant C_1 can be evaluated by application of the non-slip boundary condition at the wall: $$u(r_o)=0 \quad \Rightarrow C_1=\frac{A^*}{i^3\rho\omega}\frac{1}{J_o(\lambda r_o)}$$

Finally, the general solution is: $$u(r)=\frac{A^*}{i^3\rho\omega}\left[\frac{J_o(\lambda r)}{J_o(\lambda r_o)}-1\right]$$ or, using $$\lambda^2=\frac{i^3\omega}{\nu}$$

$$u(r)=\frac{A^*}{i\rho\omega}\left[1-\frac{J_o(r\sqrt{\frac{\omega}{\nu}}\cdot i^{3/2})}{J_o(r_o\sqrt{\frac{\omega}{\nu}}\cdot i^{3/2})}\right] \qquad (7)$$

We may now define the dimensionless **Womersley parameter alpha (α)** as $\alpha = r_o \sqrt{\frac{\omega}{\nu}} = r_o \sqrt{\frac{\omega\rho}{\mu}}$

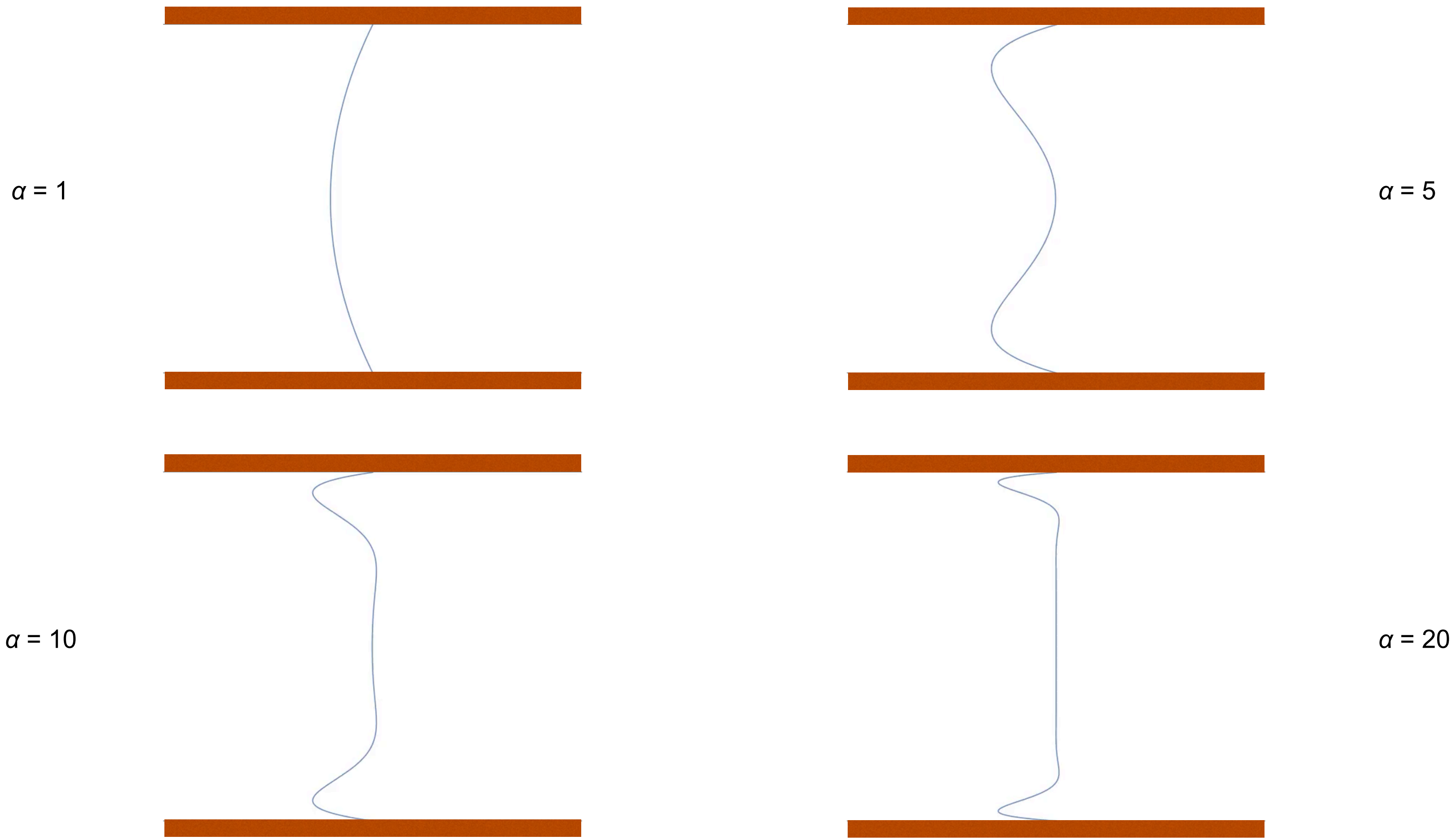
Rewrite Eq. (7) as $$u(r) = \frac{A^*}{i\rho\omega}\left[1 - \frac{J_o(\frac{r}{r_o}\alpha \cdot i^{3/2})}{J_o(\alpha \cdot i^{3/2})}\right] \qquad (8)$$

The final solution for the velocity is the real part of $v^*(r,t) = u(r)e^{i\omega t}$

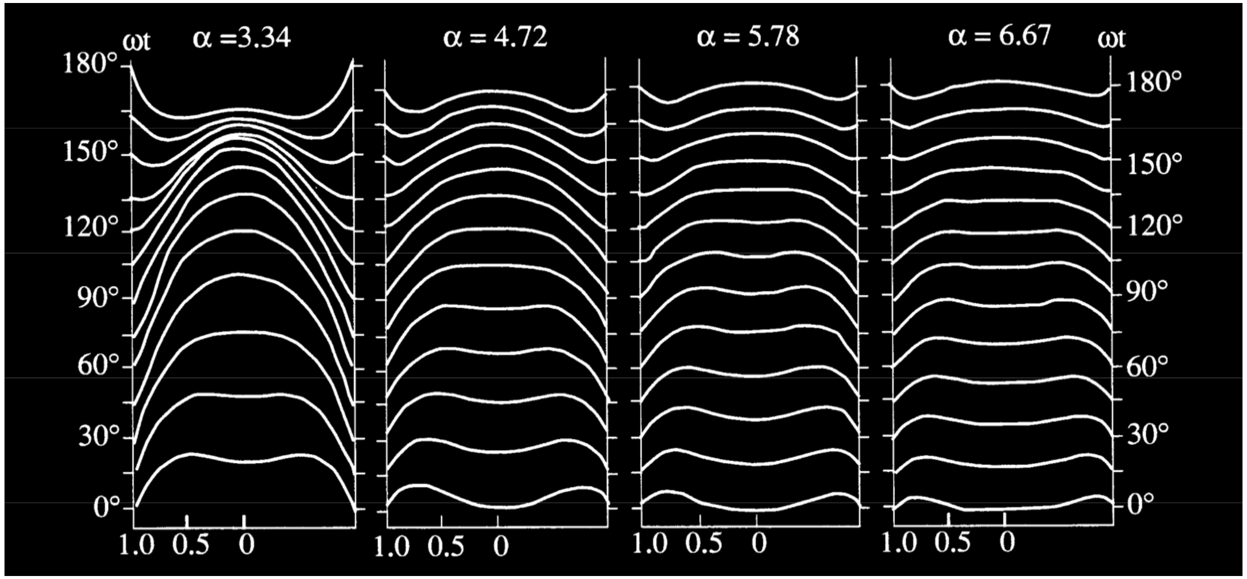
$$v(r,t) = \mathrm{Re}\left[\frac{A^*}{i\rho\omega}\left[1 - \frac{J_o(\frac{r}{r_o}\alpha \cdot i^{3/2})}{J_o(\alpha \cdot i^{3/2})}\right]e^{i\omega t}\right]$$

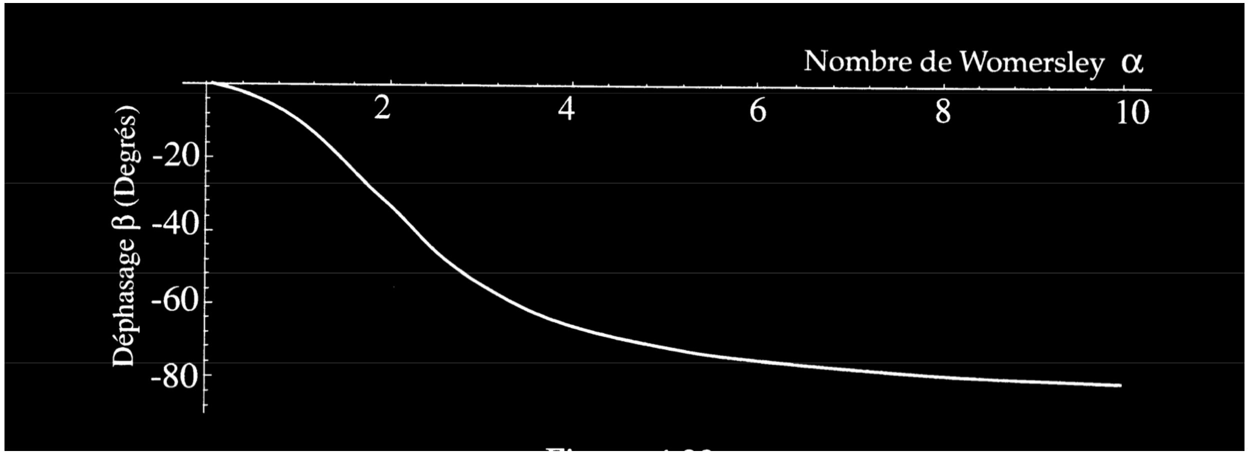
Velocity profile for a single harmonic

Pulsatile flow profiles



Velocity profiles & phase shift





Remarks:

1. The velocity profile shows that not all points along the radius move in phase

2. The phase shift, *β*, between the velocity *v(r,t)* and the pressure gradient *ΔP/l*(t) is given in the figure on the left.
3. For high Womersley numbers, i.e., for inertia-dominated flows, the phase shift tends to –90 degrees, which means that velocity lags pressure gradient by 90 degrees.

Relation of flow to pressure gradient

$$Q(t)=\int_0^{r_o} v(r,t)\cdot 2\pi r\,dr \qquad \Rightarrow Q(t)=\frac{\pi r_o^2 A^*}{i\omega\rho}\left(1-\frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2}J_0(\alpha i^{3/2})}\right)e^{i\omega t} \qquad (9)$$

Womersley named the complex term in the parenthesis 1-F$_{10}$: $1-F_{10}=1-\frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2}J_0(\alpha i^{3/2})}$

The real part of the pressure gradient is written as $\frac{\Delta P}{\ell}=M\cos(\omega t+\varphi)$

Eq. 9 can be written as: $Q(t)=\frac{\pi r_o^2}{\omega\rho}M[1-F_{10}]\sin(\omega t+\varphi)$

We express [1- F10] in terms of its modulus (M'_{10}) and phase (ε_{10}), to obtain:

$$Q(t)=\frac{\pi r_o^2}{\omega\rho}MM'_{10}\sin(\omega t+\varphi+\varepsilon_{10})$$

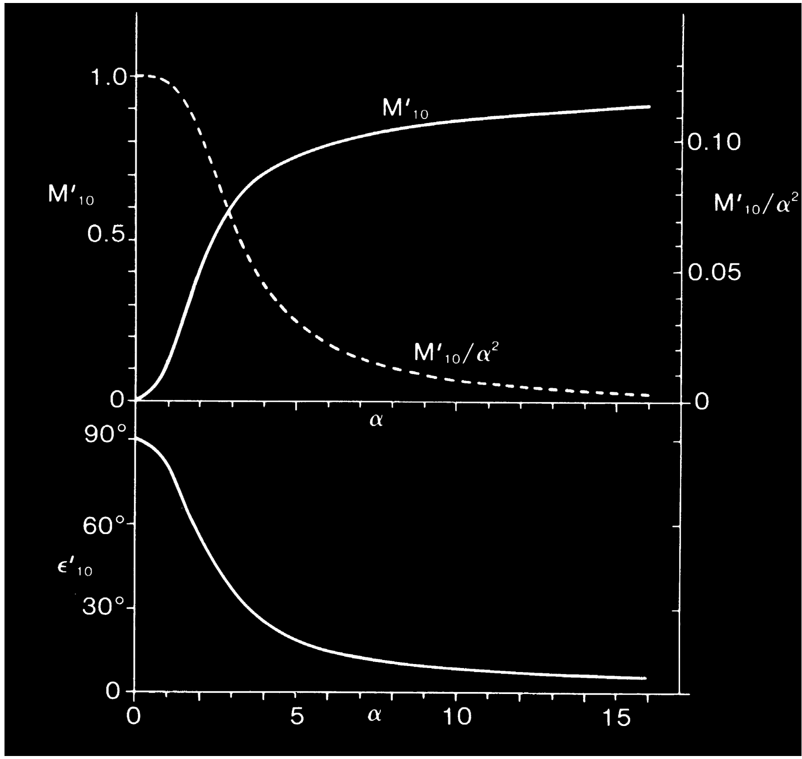
To allow comparison with Poiseuille's equation, we substitute for $\alpha^2=\frac{r_o^2\omega\rho}{\mu}$

$$Q(t)=\frac{\pi r_o^4}{\mu}M\frac{M'_{10}}{\alpha^2}\sin(\omega t+\varphi+\varepsilon_{10})$$

Note: as $\alpha\to 0$, $\frac{M'_{10}}{\alpha^2}\to\frac{1}{8}$ and $\varepsilon_{10}\to 90^o$ so that

$$Q(t)=\frac{\pi r_o^4}{8\mu}M\cos(\omega t+\varphi)$$

Poisseuille's law



Physical meaning of Womersley parameter α

$$\underbrace{\rho\frac{\partial u_i}{\partial t}}_{\text{transient inertia}} + \underbrace{\rho u_j\frac{\partial u_i}{\partial x_j}}_{\text{convective inertia}} = \underbrace{X_i}_{\text{body force}} - \underbrace{\frac{\partial p}{\partial x_i}}_{\text{pressure force}} + \underbrace{\mu\left(\frac{\partial^2 u_i}{\partial x_i^2}+\frac{\partial^2 u_i}{\partial x_j^2}+\frac{\partial^2 u_i}{\partial x_k^2}\right)}_{\text{viscous force}}$$

x-momentum (Navier-Stokes)

R = characteristic length (radius)

ω=characteristic frequency (1/characteristic time)

U = characteristic velocity

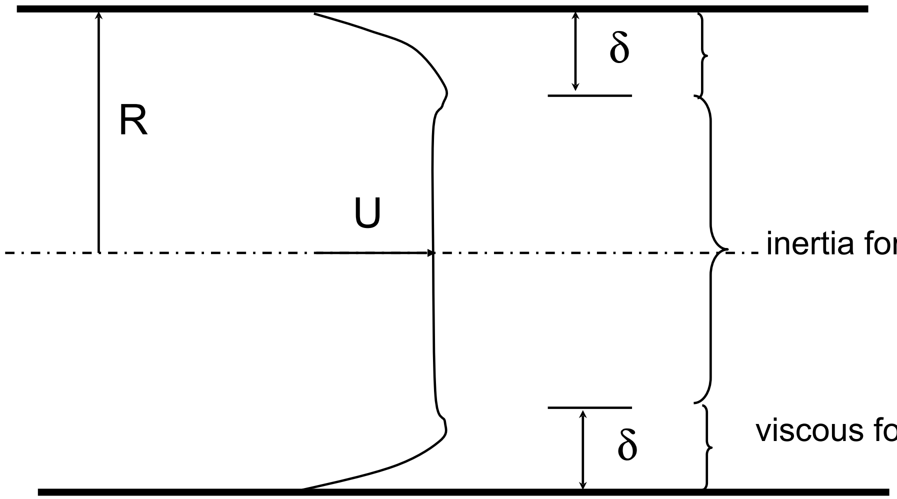
For a harmonic velocity: $u = U\cdot\sin\omega t$

Order of magnitude: $\frac{\partial u}{\partial t} = U\cdot\omega\cdot\cos\omega t \sim U\cdot\omega$

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{U}{R^2}$$

$$\frac{\textit{transient inertia force}}{\textit{viscous force}} = \frac{\rho\frac{\partial u}{\partial t}}{\mu\frac{\partial^2 u}{\partial x^2}} \approx \frac{\rho\omega U}{\frac{\mu U}{R^2}} = \frac{\omega R^2}{\frac{\mu}{\rho}} = \frac{\omega R^2}{\nu} = \alpha^2$$

Womersley parameter and viscous layer thickness



inertia forces ($\rho\omega U$) dominate

viscous forces ($\frac{\mu U}{\delta^2}$) dominate

At interface forces should be equal: $$\rho\omega U = \mu U/\delta^2 \Rightarrow \delta = \sqrt{\frac{\mu}{\rho\omega}} = \sqrt{\frac{\nu}{\omega}}$$

Ratio of tube radius to viscous layer thickness: $$\frac{R}{\delta} = R\sqrt{\frac{\omega}{\nu}} = \alpha$$

Conclusion: when $\alpha \uparrow \Rightarrow \delta \downarrow$

Pulsatile flow profiles (measurements)